Background Documents on
Risk Assessment in Engineering

Document #5

Optimization with a LQI Acceptance Criterion

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1. Introduction

This example collection covers some typical applications in structural reliability and discusses several important aspects. It is meant to be the basis by example applications for practicable acceptance criteria for structural codes. The first three examples show the influence of the interest rates in an optimization for the public or the owner, the optimal and acceptable solutions for different cost values and the importance of the coefficient of variation of the resistance and load variables. Especially in the first two examples a number of parameter studies are performed. The next example deals with the realistic design for the buckling of a reinforced concrete column. Two examples for the optimal design under reliability constraints for different load combinations of intermittent load processes will be evaluated. Finally, an example from earthquake engineering is included. The obective functions are based on a systematic recontruction policy and a constant benefit rate. Optimization is carried out by the methods described in [9] but other methods are also possible. All LQI considerations are based on [6] . Usually, the LQI-criterion is added to the cost-benefit optimization task as a constraint. In [6] it is shown that the LQI-criterion requires

$$dC_Y(\mathbf{p}) \geq -G_x k N_{PE} dh(\mathbf{p}) \tag{1}$$

with $\mathbf{p}$ a vector of design parameters. Using $dF(\mathbf{x}) = f(\mathbf{x})d\mathbf{x}$ one again solves this equation as an optimization task

$$\textbf{Minimize: } S(\mathbf{p}) = C(\mathbf{p}) + G_x k N_F h(\mathbf{p}) \tag{2}$$

because eq. (1) is the first-order optimality condition of eq. (2) if no cost-benefit optimization is performed.

From [6] one concludes that the ”societal value of a statistical life” is $G_x = \frac{g}{q}C_x$ where ”x” stands for the particular mortality reduction scheme. For constant mortality reductions Δ, a European population with $GDP \approx 25000$ PPPUS$(year 2000), the part available for risk reduction $g \approx 17500$ PPPUS$ and $q \approx 0.15$, $C \approx 45$ and $G \approx 5 Mill$ US$ if no discounting and no age-averaging. If discounting and age-averaging is performed it is $C_{d\Delta} \approx 16$ and $G_{d\Delta} \approx 1.9 Mill$ US$ if discounting and age-averaging is performed and constant mortality reductions are envisaged by a safety measure. These values then enable to compute the societal willingness-to-pay to save a statistical life by $SWTP = \frac{g}{q}C_x dm$.

The examples have taken from several sources but many of them are contained in a collection of examples in [2] . Therefore, some variations in important parameters, especially for the LQI-acceptability criterion, remain.

2. Example 1: Resistance-demand problem

The example has already been given in [5] and in [7] in somewhat different form and with different parameters. A single-mode system is considered where failure is defined if a random resistance or capacity is exceeded by a random demand. The demand is modelled as a one-dimensional, stationary marked Poissonian pulse process of disturbances (earthquakes, wind storms, explosions, etc.) with stationary renewal rate λ and random, independent sizes of the disturbances $S_i, i = 1, 2, ...$. The disturbances are assumed to be short as compared to their mean interarrival times. We study normally distributed and log-normally resistance and disturbances. The resistance has mean p and a coefficient of variation V_R. The disturbances are independent and have mean equal to unity and coefficient of variation V_S so that p can be interpreted as central safety factor. p is taken as the only optimization parameter. For failure once a disturbance occurs with normally distributed variables we have:

$$P_f(p) = \Phi\left(-\frac{p-1}{\sqrt{(pV_R)^2 + V_S^2}}\right) \tag{3}$$

In this model both resistances and disturbances can have negative values which usually is incorrect from a physical point of view. Therefore, we truncate the distributions at zero and a more complicated formula for the failure probability results:

$$P_f(p) = \frac{1}{\Phi(\frac{1}{V_S})}\frac{1}{\Phi(\frac{1}{V_R})}\int_0^\infty \frac{1}{\sqrt{2\pi}V_S}\varphi(\frac{s-1}{V_S})\int_0^s \frac{1}{\sqrt{2\pi}V_R p}\varphi(\frac{r-p}{V_R p})drds \tag{4}$$

For the log-normal case we have:

$$P_f(p) = \Phi\left(-\frac{\ln\left\{p\sqrt{\frac{1+V_S^2}{1+V_R^2}}\right\}}{\sqrt{\ln\left((1+V_R^2)(1+V_S^2)\right)}}\right) \tag{5}$$

An appropriate objective function for systematic reconstruction which will be maximized, is then given by

$$Z(p) = \frac{b}{\gamma} - C(p) - (C(p) + H_M + H_F)\frac{\lambda P_f(p)}{\gamma} \tag{6}$$

where $C(p) = C_0 + C_1\, p^a$ and H_M the initial material cost.

The acceptability criterion can be written as

$$\frac{d}{dp}C(p) \geq -G_\Delta k N_{PE}\frac{d}{dp}\lambda P_f(p) \tag{7}$$

The parameter assumptions are: $C_0 = 10^6$, $C_1 = 10^4$, $a = 1.25$, $H_M = 3\,C_0$, $V_R = 0.2$, $V_S = 0.3$ and $\lambda = 1\ [1/year]$. The LQI-data are $e = 77$, $g = 25000$, $C_\Delta = 25$, $q = 0.16$, $kN_{PE} = 10$ so that $H_F \approx 7 \cdot 10^6$ and $G_\Delta(\rho, \delta) \approx 5 \cdot 10^6$. Monetary values are in US$. Optimization will first be performed for the public. Therefore, compensation cost are included in the objective function. The benefit rate is $b = 0.02\,C_0$ and the interest rate is $\gamma = 0.0185$. For the failure probability model eq. (3) one finds the optimum for $p^* = 5.54$ and a limiting value of $p_{\lim} = 3.86$. If one uses the model in eq. (4) the optimum is at $p^* = 3.00$ and the limiting criterion eq. (7) results in $p_{\lim} = 2.49$. Finally, for the model in eq. (5) the optimum is at $p^* = 4.40$ and has a limit of $p_{\lim} = 3.45$. The three objective functions are shown in figure 1. One concludes from this study that the stochastic models for the failure function must be as realistic as possible. All subsequent investigations are based on log-normal distributions.

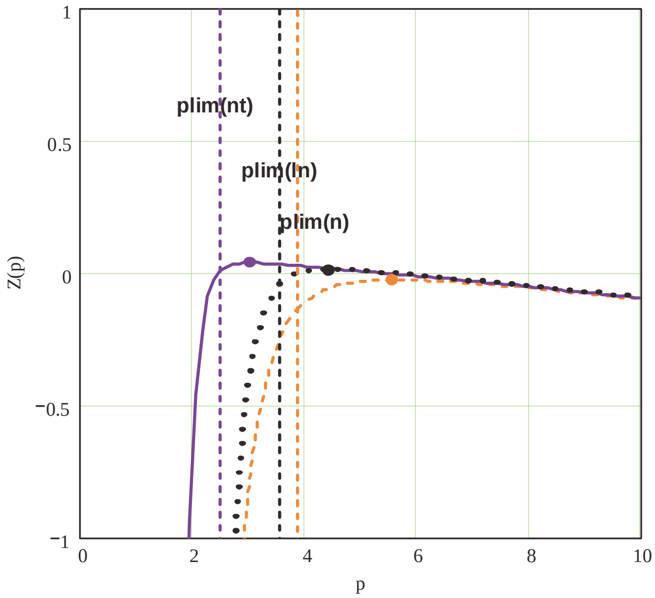

Figure 1: Objective functions and limiting criteria for the failure models in eq. (1) (dashed line), (2) (solid line) and (3) (dotted line)

The influence of the cost C_1 on the optimal and acceptable solution is shown in figure 2, where C_1 varies from 1000.0 to 100000.0 and the corresponding optimal and acceptable failure rates are given. For $C_1 > 12000$ the objective function $Z(p^*)$ is negative.

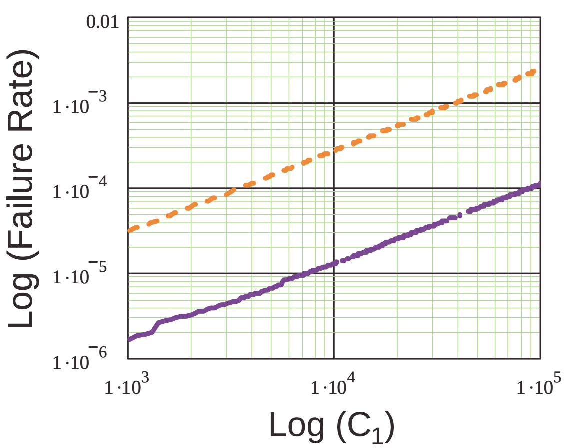

Figure 2: Failure rates for various cost C_1. Dashed lines correspond to acceptable, solid lines to optimal results.

The optimal solution is acceptable for all cost values C_1. It is most important to ensure that all necessary investments into life saving in order to guarantee the safety for human life and limb are done and the resulting failure rate of a technical facility is acceptable . This is already fulfilled if the acceptability criterion derived from the LQI is fulfilled.

The owner uses some typical value $b = 0.07 \cdot C_0$. Compensation cost (Life saving cost) are not included in the optimization. The objective of the owner is in general to maximize the benefit from an investment. Therefore, he uses a higher interest rate $\zeta = \gamma_{owner}$ given by the financial market. But he is still bound to build safe structures. This is only guaranteed if the LQI-criterion which may includes the ratio ζ/γ_{public} of the economic and the public interest rate γ_{public} given by the characteristics of a society is fulfilled. The corresponding failure rates of the optimization for various interest rates γ_{owner} are given in figure 3 with $kN_F = 50$. In general for a long period the ratio ζ/γ_{public} is or is close to 1. For shorter service times or credit periods a different value is possible. Therefore, the acceptable failure rate is shown in figure 8 for $\zeta/\gamma_{public} = 0.7$, $\zeta/\gamma_{public} = 1.0$ and $\zeta/\gamma_{public} = 1.3$.

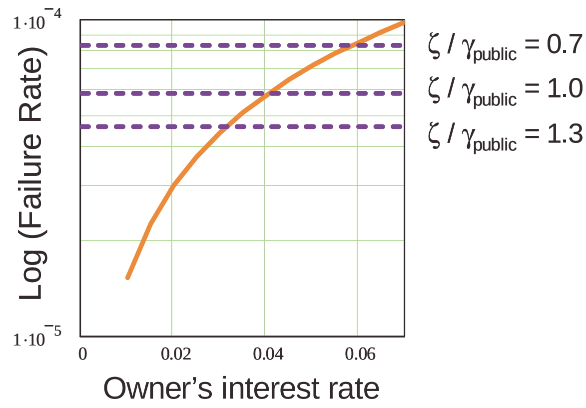

Figure 3: Failure rates for various interest rates of the owner. Dashed lines correspond to acceptable, solid lines to optimal results.

It can be seen, that the LQI-criterion (7) becomes active for increasing interest rates γ_{owner} depending on the ratio ζ/γ_{public}. In that case the owner must use the acceptable solution. Further calculations show, that for $kN_F > 50$ the LQI-criterion becomes permanently active. $kN_F < 50$ makes the optimal solution always acceptable.

For the standard case the ratio H/C_0 is varied between 1 and 10 in figure 4, a range which covers most applications. It is seen that increasing H by an order of magnitude decreases the optimal failure rate by roughly half an order of magnitude.

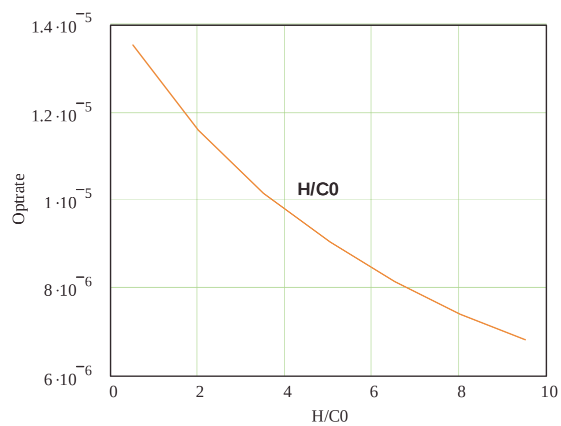

Figure 4: Optimal failure rate with increasing ratio H/C_0

This example also allows to derive risk-consequence curves by varying the number of fatalities in an event. With the same data as before but $SLSC = 7 \cdot 10^5$ and $G_x(\rho, \delta) = 4 \cdot 10^6$ for $N_F = 1$

we first vary the cost effectiveness of the safety measure (see figure 5). Here, only the ratio C_1/C_0 is changed. The upper bounds (solid lines) are derived from eq. (7) and the lower bounds (dashed lines) corresponds to the societal optimum according to eq. (6) ($b_S = 0.02C_0$, $\gamma_S = 0.0185$).

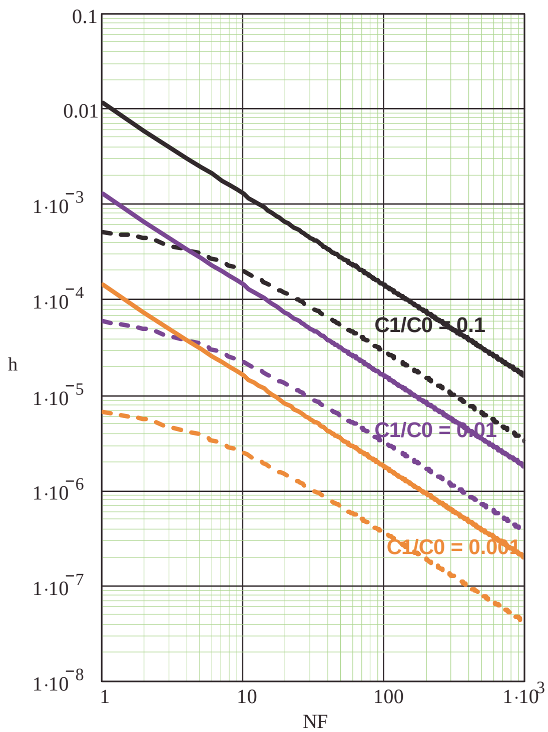

Figure 5: Acceptable failure rate over number of fatalities for different C_1/C_0 for first example. Dashed lines correspond to optimal solution for the public.

Most realistic is probably a ratio of $C_1/C_0 = 0.001$. The failure rate of approximately 10^{-4} per year for $N_F = 1$ corresponds well with the "controllable" crude mortality of the same magnitude. Some more discussion is provided in the next example.

3. Example 2: Resistance-demand problem - variable discounting

The same failure model as in example 1 (eq. (5)) is used with variable discounting. An appropriate objective function then is:

$$Z(p) = \frac{b}{C_0}\int_0^\infty e^{-\int_0^t \gamma(\tau)d\tau} dt - \left(1 + \frac{C_1}{C_0}p^a\right) - \left(1 + \frac{C_1}{C_0}p^a + \frac{H_M}{C_0} + \frac{H_F}{C_0}\right)\int_0^\infty e^{-\int_0^t \gamma(\tau)d\tau} g(t,p)dt \qquad (8)$$

with

$$g(t,p) = \sum_{m=1}^{\infty} P_f(p) f_m(t)(1 - P_f(p))^{m-1} \qquad (9)$$

$$f_m(t) = \frac{\lambda(\lambda t)^{m-1}}{\Gamma(m)} \exp\left[-\lambda t\right] \qquad (10)$$

and

$$\gamma(t) = \epsilon\delta + \rho_{\max} \exp(-at);\ \epsilon\delta = 0.02, a = 0.013 \tag{11}$$

Here, we use the exact computation formula for $g(t,p)$ and not the approximation proposed in [7] . In eq. (8) the first term is the benefit derived from the existence of the facility. A constant benefit rate b is assumed and a time-depending discount rate $\gamma(t)$. The second term is the erection cost where C_0 is the constant part and $C_1 p^a$ the part depending on the design parameter p. The third term are the damage cost. Because systematic reconstruction is chosen the cost term includes reconstruction cost, cost H_M due to physical damage, and the compensation cost for human lives H_F. Reconstruction times are assumed negligibly small. The details are discussed in [7] .

The acceptability criterion has the form:

$$\frac{d}{dp}\left(C_0 + C_1 p^a\right) \geq -G_{\Delta \bar{\ell}} k N_{PE} \frac{d}{dp}\left(\lambda P_f(p)\right) \tag{12}$$

Some more or less realistic, typical parameter assumptions are: $C_0 = 10^6$, $C_1 = 10^4$, $a = 1.25$, $H_M = 3 \cdot C_0$, $V_R = 0.2$, $V_S = 0.3$, $b = 0.05 C_0$ and $\lambda = 1\ [1/year]$. A constant benefit rate of $b = 0.05 C_0$ is chosen. The LQI-data is $e = 77$, $GDP = 25000$, $g = 17500$, $q = 0.15$. Also there is $N_{PE} = 100, k = 0.1$ so that $H_F = SLSC\ kN_{PE} = 5.4 \cdot 10^6$ and $G_\Delta k N_{PE} = 5.0 \cdot 10^7$, respectively. The value of N_{PE} is chosen relatively large for demonstration purposes. Also, the values of H_F and $G_\Delta k N_{PE}$ are conservatively chosen. Monetary values are in US$.

From eq. (8) one concludes that there is a maximum discount rate in order to render it positive. Projects whose objective is negative at the optimum do not make sense. Keeping $\epsilon\delta = 0.02$ constant one can determine the maximum time preference rate from

$$\rho_{\max} = \text{ solution of } \left(Z(p_{opt}, \rho) = 0\right) \tag{13}$$

which is $\rho_{\max} = 0.035$.

Performing the optimization with the (maximum) discount rate $\rho_{\max}$ gives $p_{opt} = 4.04$ ($r(p_{opt}) = 3.0 \cdot 10^{-5}$) leaving the objective function positive but close to zero. Criterion (12) requires $p_{\lim} = 3.61$ ($r(p_{\lim}) = 1.1 \cdot 10^{-4}$). It is interesting to see that in this case one can do better in adopting the optimal solution rather than just realizing the facility at its acceptability limit (see fig. 6). If, however, compensation cost are omitted from eq. (8) then criterion (12) is situated very close to the optimum. Increasing the discount rate moves the optimum insignificantly to the left.

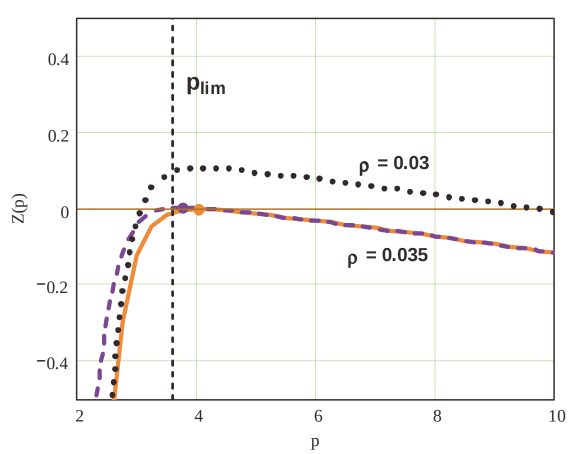

Figure 6: Objective function and acceptability limit for capacity-demand example for two

discount rates (solid line: with compensation cost, dashed line: without compensation cost)

For smaller ρ (or larger benefit rate) the maximum of the objective function (with compensation cost) is well above zero as shown in fig. 6 (dotted line). The location of the optimum varies very little with the discount rate ρ as shown in fig. 7.

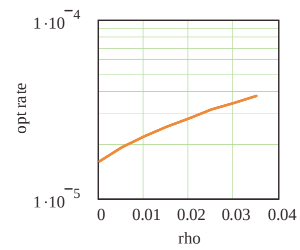

Figure 7: Location of optimum as a function of ρ

The stochastic model and the variability of capacity and demand also play an important role for the magnitude and location of the optimum as well as on the position of the acceptability limit. The specific marginal cost (rate of change) of a safety measure and its effect on a reduction of the failure rate are equally important as pointed out already in [5] . This is shown in fig. 8 where the acceptable failure rates are plotted over the parameters C_1 and V_R for $G_{\Delta\bar{\ell}} k N_{PE} = 5.0 \cdot 10^6$,i.e. $kN_{PE} = 1$, and $C_0 = 10^6$.

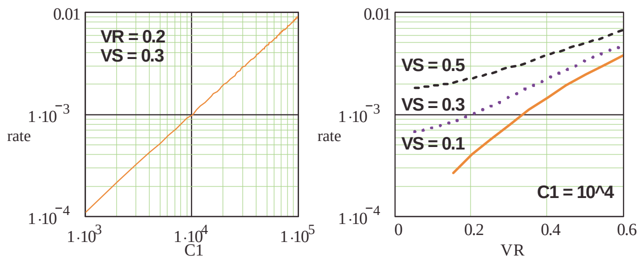

Figure 8: Location of acceptable failure rate as function of marginal cost for safety and coefficients of variation

This example also allows to vary the upper value m in eq. (9). It gives an impression about the effect to consider only a limited number of reconstructions on the optimum. As seen in figure 9, the effect can be remarkable.

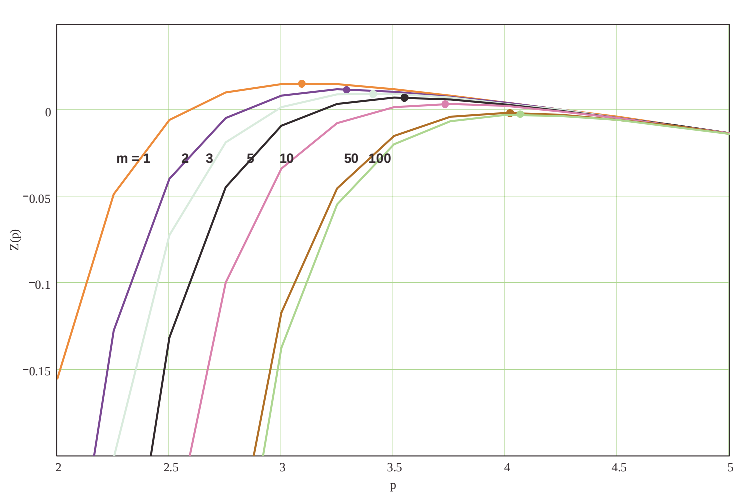

Figure 9: Optimum solutions for different numbers of reconstructions

For this example risk-consequence curves can also be computed by varying the number of fatalities in an event. For the same data as before we first vary the cost effectiveness of the safety measure. Only the ratio C_1/C_0 is changed. The upper bounds (solid lines) are derived from eq. (12). The lower bounds (dashed lines) correspond to the optimum according to eq. (8) (see left hand side of fig. 10). $\rho_{\max}$ also must vary (decrease) with the failure consequences (number of fatalities N_F) for a given ratio C_1/C_0. This is taken into account on the right hand side of fig. 10 indicating that the area between solid and dashed lines broadens for very high failure consequences.

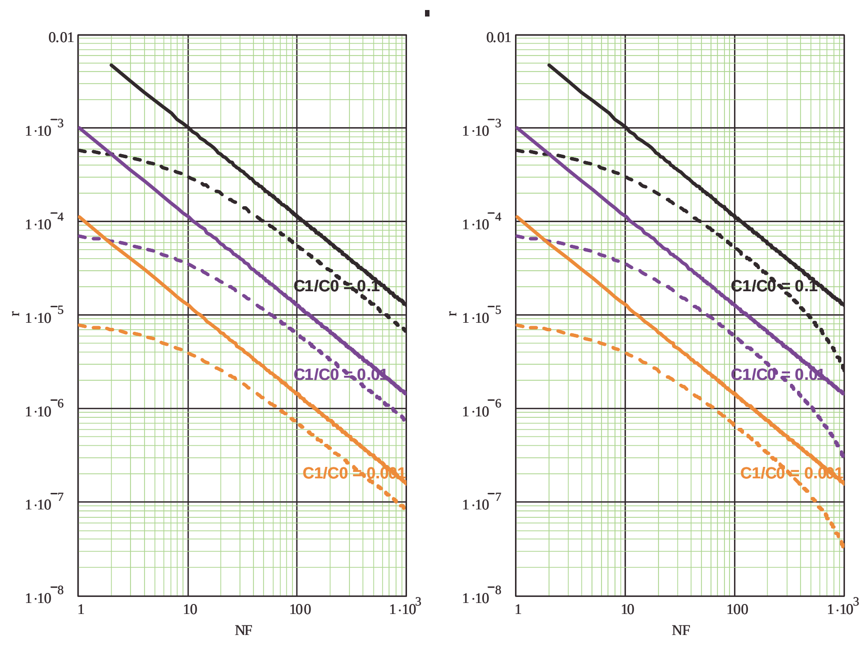

Figure 10: Risk-Consequence curves for various ratios C_1/C_0 ($V_R = 0.2$, $V_S = 0.3$, $\rho_{\max} = 0.035$)

Again, normal structures have a ratio of around $C_1/C_0 = 0.001$. A ratio of $C_1/C_0 = 0.01$ or higher may apply for earthquake resistant structures. Note that in these figures the failure rate is given by λP_f and the number of fatalities is given by $N_F = kN_{PE}$. Therefore, these figures cover the full range of λ and P_f and k and N_{PE}, respectively. The curves in fig. 10 are not classical F-N-curves seen frequently in the literature where the exceedance probability instead of the failure rate is plotted over the number of fatalities. The acceptable risk-consequence curves show an almost perfect dependence of the type $r_{acc}(N_F) = \frac{K}{N_F}$ with $K = 10^{-3}$ (case with large variabilities and/or costly safety measures), 10^{-4} (standard case) and, may be, 10^{-5} (for small variabilities and inexpensive safety measures). Reasonable variations in the other parameters change these constants at most by a factor of 2 to 4 as seen from fig. 11 where the extremes of the demographic constant are shown. In particular, using $G_{\Delta\bar{\ell}} = 1.9Mill$ PPPUS$ would move the curves in fig. 11 upwards be a factor of 3.

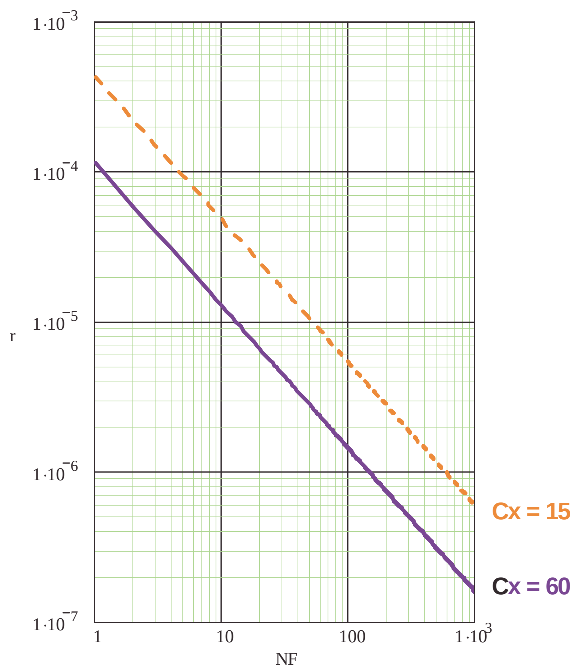

Figure 11: Risk consequence curve with two extreme of the demographic constant for $C_1/C_0 = 0.001$.

The demographic constant $C_x \approx 15$ corresponds to discounted life expectancies with age-averaging. $C_x \approx 60$ corresponds to very old populations without any discounting and age-averaging. The curves also indicate that larger GDP's lead to smaller acceptable risks and larger q's can diminish them.

4. Example 3: Bending of a rectangular reinforced concrete beam

The benefit b derived from the structure and the interest rate γ vary with respect to the publics and the owners interest. An appropriate objective function for systematic reconstruction and stationary Poissonian disturbances is already given by eq. (6). $C(p) = C_0 + (400\ [CU/m^3] \cdot w \cdot d + 1000\ [CU/m^3] \cdot a_s) \cdot 5\ [m]$ are the construction cost with $C_0 = 1000$ and the reinforcement area $a_s\ [cm^2]$, width $w\ [cm]$ and effective depth $d\ [cm]$ of the cross section. Two constraints are defined on the design parameter (a_s, w, d): a maximum allowable reinforcement area $a_s \leq 4\%\ w \cdot d$ and a lower bound for the cross-sectional area $w \cdot d \geq 0.01\ [m^2]$. The stochastic characteristics of the random variables are given by

Variable	Distr.	Mean/ St.deviat.
Yield stress T_s $\left[\frac{N}{mm^2}\right]$	N	360/36
Conc. strength T_c $\left[\frac{N}{mm^2}\right]$	LN	40/6
Appl. bend. mom. M_b $[MNm]$	Gumbel	0.05/0.003
Model uncert. K $[-]$	Rect.	0.5/0.667

The limit state function is determined by the simplified model

$$g(\mathbf{x},\mathbf{p}) = \left(1 - K\frac{a_s T_s}{w\,d\,T_c}\right) a_s\,d\,T_s \;-M_b \tag{14}$$

and the acceptability criterion by

$$\bigtriangledown_{\mathbf{p}}\, C(\mathbf{p}) \geq -G_{\Delta\bar{\ell}} k N_{PE} \bigtriangledown_{\mathbf{p}} (\lambda P_f(\mathbf{p})) \tag{15}$$

The LQI-data are $e = 77$, $g = 25010$, $C_{\Delta\bar{\ell}} = 16$, $q = 0.16$, $kN_{PE} = 0.1$, $\lambda = 1[1/year]$ and therefore $H_F = 88000$ and $G_{\Delta\bar{\ell}} kN_{PE} = 1.9 \cdot 10^5$. The direct material damage cost are $H_M = 3 \cdot C_0 = 3000$. $kN_F = 0.1 < 1$ takes account of the fact, that failure is ductile.

Optimization will first be performed for the public. The interest rate $\gamma = 0.018$ and the benefit $b = 0.02C_0 = 20$ are chosen. The optimal solution is then $(a_s^*,\, w^*,\, d^*) = (14.45, 15.0, 24.09)$, the corresponding failure rate $2.51 \cdot 10^{-7}$and $Z(\mathbf{p}^*) = 30.34$. The LQI-criterion (15) is already fulfilled for $(a_{s,\text{lim}}^*, w_{\text{lim}}^*, d_{\text{lim}}^*) = (13.41, 15.0, 22.37)$ with failure rate $1.55 \cdot 10^{-5}$ and $Z(\mathbf{p}_{\text{lim}}^*) = -42.0$. It is interesting to note that for $kN_F = 1$ the limiting failure rate is one order of magnitude higher. Here, the optimization for the public gives also a more economic result than the acceptable solution and is preferable. In this example the life saving cost (compensation cost) H_F dominate the optimization compared to the low cost H_M and low marginal cost C_1. Therefore, the solution is sensitive to the value of the parameter kN_{PE}. This is demostrated in figure 12. If a large number of fatalities is expected the acceptable failure rate decreases. This is, however, not very likely because failure of under-reinforced beams is ductile and associated with distinct warnings by large deflections.

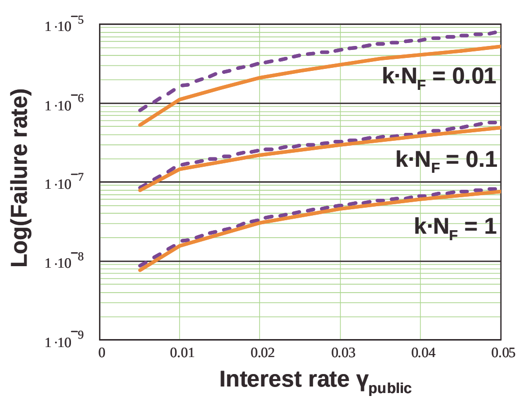

Figure 12: Optimal and acceptable solutions for parameter values $k \cdot N_F$. Dashed lines correspond to acceptable, solid lines to optimal results.

For a maximum admissible interest rate ([4]) $\gamma_{\max} = 0.0185$ the objective function is about $Z(\mathbf{p}^*) \approx 0$ with $(a_s^*,\, w^*,\, d^*) = (14.44, 15.0, 24.07)$. If $\gamma_{\max}$ is chosen the benefit derived from a structure is balanced against all cost including initial, life saving and maintenance cost, which is preferable from a society's point of view. The objective of the owner is in general to benefit from an investment. Typical values are then $b = 0.07 \cdot C_0 = 70$ and $\gamma = 0.05$. The optimization yields without including failure cost to $(a_s^*, w^*, d^*) = (13.37, 15.0, 22.29)$, a corresponding failure rate of $1.86 \cdot 10^{-5}$ and $Z(\mathbf{p}^*) = 324.9$. But the criterion (15) is not fulfilled. Therefore, the solution is not acceptable. It requires at least $(a_{s,\mathrm{lim}}^*, w_{\mathrm{lim}}^*, d_{\mathrm{lim}}^*) = (13.41, 15.0, 22.37)$ with failure rate $1.55 \cdot 10^{-5}$ and $Z(\mathbf{p}_{\mathrm{lim}}^*) = 324.7$.

It can be seen, that in this example the optimal solution for the public gives a more economic result than the limit point derived from criterion (15) which is not active. The example is one where it is comparatively cheap to improve safety.

5. Example 4: Slender steel column

The next example is a steel column with I-section and flange breadth μ_b, flange thickness μ_d and height of steel profile μ_h. The design parameter μ_d and μ_h will be changed in the cost optimization. They are mean values of appropriate stochastic variables.

The objective function for systematic reconstruction and a stationary Poissonian renewal process of disturbances is given in eq. (6). $C(p) = C_0 + C_1\ (2\mu_b\mu_d + 10\mu_h)\ s\ 10^{-9}$ are the construction cost of the steel column with length $s = 9500$ [mm], initial cost $C_0 = 10000\ [CU]$ and $C_1 = 10000\ [CU/m^3]$. $\lambda = 10.1\ [1/year]$ is a stationary renewal rate. The stochastic characteristics of the random variables are given by

Stochastic variable	Distr.	Mean/St.deviat.
Yield stress F_S $\left[\frac{N}{mm^2}\right]$	N	500/25
Load P_1=P_2 [N]	N	800000/150000
Load P_3 [N]	LN	800000/200000
Web thickness t [mm]	N	10/0.5
Flange Breadth B [mm]	LN	μ_b = 300/3.0
Flange Thickness D [mm]	LN	μ_d/2.0
Height of profile H [mm]	LN	μ_h/5.0
Initial deflection F_0 [mm]	N	m_{F_0}/σ_{F_0}
Youngs Modulus E $\left[\frac{N}{mm^2}\right]$	N	210000/4200

where the mean value m_{F_0} of F_0 is defined by

$$m_{F_0} = \frac{1}{20}\sqrt{\frac{0.5 \cdot B \cdot D \cdot H^{\,2} + \frac{t \cdot H^{\,3}}{12}}{2 \cdot B \cdot D + t \cdot H}} + \frac{s}{500}$$

and $\sigma_{F_0} = 0.3 \cdot m_{F_0}$. The limit state function for failure due to instability can be written approximately as

$$g(\mathbf{x}, \mathbf{p}) = F_S - P \cdot \left(\frac{1}{A_S} + \frac{F_0}{M_S} \cdot \frac{\epsilon_b}{\epsilon_b - P}\right) \tag{16}$$

where the auxiliary functions are defined by

$$\begin{aligned}
P &= P_1 + P_2 + P_3 \\
A_S &= 2BD \text{ (area of section)} \\
M_S &= BDH \text{ (modulus of section)} \\
M_i &= \tfrac{1}{2}BDH^2 \text{ (moment of inertia)} \\
\epsilon_b &= \tfrac{\pi^2 E M_i}{s^2} \text{ (Euler buckling load)}
\end{aligned}$$

The acceptability criterion is given in eq. (15) and the LQI-data are $e = 77$, $g = 25010$, $C_{\Delta\bar{\ell}} = 16$, $q = 0.16$, $kN_{PE} = 1$, and therefore $H_F = 8.8 \cdot 10^5$ and $G_{\Delta\bar{\ell}} kN_{PE} = 1.9 \cdot 10^6$. The direct cost are $H_M = 3 \cdot C_0 = 30000$. The objective function $Z(\mathbf{p})$ including the cost H_F for the public with $b = 200$ and $\gamma = 0.0172$ is shown in figure 13.

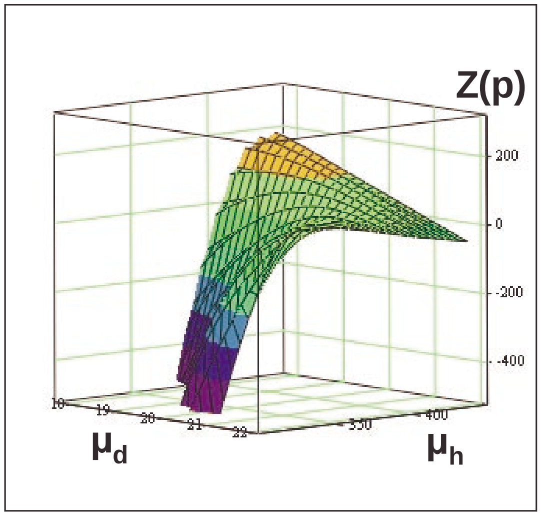

Figure 13: Objective function of the steel column.

An optimal curve can be found with respect to μ_d and μ_h (see figure 13). The LQI criterion is not active in an optimization for the public. The corresponding failure rates are between $2.8 \cdot 10^{-7}$ for $\mu_d = 18.0$, $\mu_h = 421.9$, $Z_{\max}(\mathbf{p}) = 186$ and $1.9 \cdot 10^{-7}$ for $\mu_d = 22.0$, $\mu_h = 333.5$, $Z_{\max}(\mathbf{p}) = 46.95$.

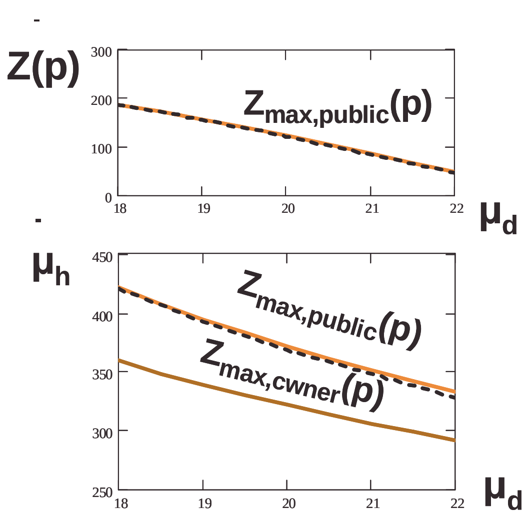

Figure 14: Optimal objective functions for the steel column. Dashed lines correspond to acceptable, solid lines to optimal results.

Additionally optimization for the owner is done in figure 14 with benefit $b = 700$ and interest rate $\gamma_{owner} = 0.06$. The curve is unacceptable compared with the solution derived from the LQI-criterion (15) if $\mu_d < 18.04$ or $\mu_h > 355.9$. Then, the owner must observe the acceptability criterion.

6. Example 5: Reinforced concrete column

This example is based on a detailed study in [3] . According to the design rules of Eurocode 1 and 2 the buckling of a reinforced concrete column of a multi-story building given in figure 15 is considered.

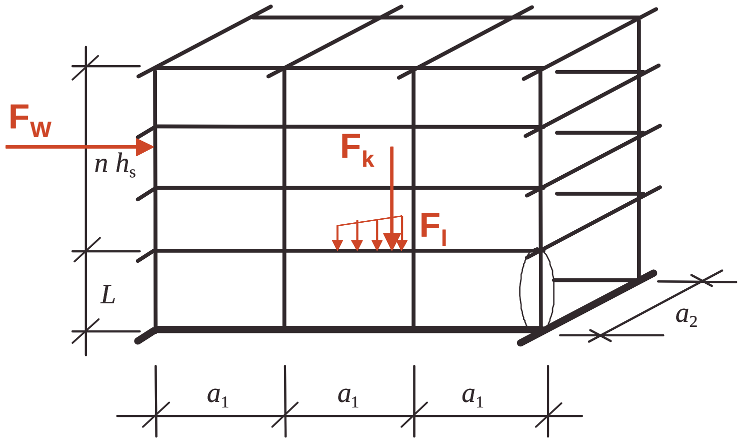

Figure 15: Multi-storey building with loads.

The design parameter is the reinforcement area A_s of the square shape $b \times h$ of the column cross section. A_s should satisfy the condition $0.003\ b\ h < A_s < 0.08\ b\ h$. The limit state function is expressed as the difference between the resistance bending moment M_R and the actual bending moment M about the cross section centre

$$G(\mathbf{x}, \mathbf{p}) = \xi_R\, M_R - \xi_E\, M \tag{17}$$

where the model uncertainties ξ_R and ξ_E are normally distributed with mean 1.1 and 1.0 respectively and coefficient of variation $V = 0.1$.

The axial force N of the column is considered as the following sum of the single axial forces:

$$N = N_W + N_{imp} + N_{Wind}$$

where N_W is the axial force due to self weight, N_{imp} is the axial force due to imposed load and N_{Wind} is the axial force due to wind action (positive values are accepted for compressive forces). The values are calculated as

$$N_W = (n+1)a_1 a_2 t \rho_c / 2$$

where ρ_c is the weight of the concrete per unit volume and n is the number of storeys with height $h_s = 3\,m$ above the considered column.

The imposed load from n storeys is

$$N_{imp} = n a_1 a_2 p_{imp}/2$$

where p_{imp} is given as sum of long term p_l and short term p_s imposed load.

N_{Wind} is given as

$$N_{Wind} = (\frac{1}{2}(L + nh_s)^2 a_2 C_p G p_{wind} - 4M_0)/(3a_1)$$

where L is the height of the column, C_p the shape factor, G the gust factor and p_{wind} the wind pressure. The first order bending moment M_0 at the bottom of the column is assumed to be caused only by wind action and is approximately given as

$$M_0 = L\left[C_p G p_{wind}(L + nh_s)a_2\right]/8$$

The actual bending moment M is then determined by

$$M = M_0 + N(e_a + e_2)$$

where $e_a = \zeta L/2$ is the additional eccentricity taking into account geometric imperfections with initial sway ζ. The second order eccentricity e_2 taking into account deformations of the column is given as

$$e_2 = 0.1K_1 L^2(2K_2(F_s/E)/(0.9(h - d_1)))$$

where $E = 200000\ N/mm^2$ is the modulus of elasticity and d_1 the distance of the reinforcing bars from the edge. The factor K_1 is calculated as

$$K_1 = \begin{cases} 0, \text{ if } \lambda \leq 15 \\ \frac{\lambda}{20} - 0.75, \text{ if } 15 < \lambda < 35 \\ 1, \text{ if } \lambda \geq 35 \end{cases}$$

with $\lambda = L/h/3.4641$.

K_2 is given by

$$0 \leq K_2 = (N_u - N)/(N_u - N_{bal}) \leq 1$$

where the ultimate capacity N_u of the cross section is

$$N_u = \alpha bhF_c + A_s F_s$$

with α as a reduction factor taking account of long term effects on the compressive strength, F_c the concrete strength and F_s the yield strength.

The balanced axial force which maximizes the ultimate moment of the cross section is taken for symmetrical reinforcement as

$$N_{bal} = \alpha bhF_c/2$$

The limit state function is then given using a simplified formula:

$$\begin{aligned} (A_s F_s(h - d_1 - d_2)/2 + hN(0.5 - N/(2\alpha hbF_c)))\xi_R - \xi_E M &> 0, \text{ if } N < N_{bal} \\ K_2(A_s F_s(h - d_1 - d_2)/2 + \alpha bh^2 F_c/8)\xi_R - \xi_E M &> 0, \text{ if } N \geq N_{bal} \end{aligned}$$

where d_1, d_2 is the distance of the reinforcing bars from the edge.

This mechanical model is in various ways an approximation which, however, is fully sufficient for the purpose. The characteristics of the stochastic variables are given in the following table

Stochastic variable	Distr.	Mean/St.deviat.
Concrete strength F_c $\left[\frac{N}{mm^2}\right]$	LN	$30/5$
Yield strength F_S $\left[\frac{N}{mm^2}\right]$	LN	$560/30$
Reduct. Factor α	Rect.	$0.8/0.95$
Width of section $b[m]$	N	$b/0.005$
Height of section $h[m]$	N	$h/0.01$
Distance of bars $d_1, d_2[m]$	N	$0.05/0.01$
Initial Sway ζ $[rad]$	N	$0/0.003$
Model Uncert. load $\xi_E[-]$	N	$1.0/0.1$
Model Uncert. Resist. $\xi_R[-]$	N	$1.1/0.11$
Weight ρ_c $\left[\frac{MN}{m^2}\right]$	N	$0.024/0.00192$
Shape factor C_p $[-]$	N	$1.3/0.13$
Gust factor G $[-]$	GUM	$2.5/0.25$

A detailed description of all random variables can be found in [3] . The loads are modelled as rectangular wave renewal processes with Gumbel distribution for the amplitudes. It is distinguished between wind pressure with mean 0.00035 $[MN/m^2]$, standard deviation 0.00006 and jump rate $\lambda = 10$ $[year]$, long term imposed load with mean 0.0006 $[MN/m^2]$, standard deviation corresponding to the random point-in time distribution and jump rate $\lambda = 1/7$ $[1/year]$ and short term imposed load with mean 0.0002 $[MN/m^2]$, standard deviation corresponding to the distribution of 24 hours maximum and jump rate $\lambda = n$ $[1/year]$ ($n =$ number of storeys).

An objective function for systematic reconstruction and Poissonian failures is

$$Z(\mathbf{p}) = \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H)\frac{\nu^+(\mathbf{p})}{\gamma} \tag{18}$$

where $C(\mathbf{p}) = C_0 + (340\ [CU/m^3]\, b\, h + 5900\ [CU/m^3]\, a_s)\, L\ [m]$ with $C_0 = 300$ and L the length of the column. The cost H contains failure cost H_F as well as cost H_M for physical damage in case of failure.

The acceptability criterion can be written as

$$\bigtriangledown_{\mathbf{p}} C(\mathbf{p}) \geq -G_{\Delta\bar{\ell}} k N_{PE} \bigtriangledown_{\mathbf{p}} \left(\nu^+(\mathbf{p})\right) \tag{19}$$

Optimization will be performed for the public with benefit $b = 0.25$ and interest rate $\gamma = 0.0175$. The LQI-data are the same as for the steel column example so that $H_F = 8.8 \cdot 10^5$ and $G_{\Delta\bar{\ell}} k N_{PE} = 1.9 \cdot 10^6$ for $kN_{PE} = 1.0$. The cost H_M for physical damage are set to $H_M = 0$ and $H_M = 2.0 \cdot 10^6$ in two different optimization runs in order to express the influence on the optimal and acceptable result.

The upper bound of the time variant failure probability in the stationary case is determined by $P_f(\mathbf{p},t) = P_f(\mathbf{p}, t_1) + (t_2 - t_1)\nu^+(\mathbf{p})$. The equivalent beta values are derived from the upper bound solution. In the following table the optimal reinforcement area is calculated for various study cases, where the column length, the cross section, the transversal distance a_1 and the longitudinal distance a_2 to the surrounding columns and the number of storeys in the building is varied. A life time $(t_2 - t_1) = 50$ years is assumed

Study case	Distance a_1 [m]	Distance a_2 [m]	Cross section $b \times h$ [m×m]	Optim. reinf. area $A_s 10^4$ $[m^2]$	equ. beta value $\left[\frac{1}{50\text{ years}}\right]$	failure rate $\left[\frac{1}{\text{year}}\right]$	Optim. reinf. area $A_s 10^4$ $[m^2]$	equ. beta value $\left[\frac{1}{50\text{ years}}\right]$	failure rate $\left[\frac{1}{\text{year}}\right]$
				$H_M = 0$			$H_M = 2.0 \cdot 10^6$		
1	5	5	0.35×0.70	9.18	3.52	3.9×10^{-6}	10.05	3.63	2.6×10^{-6}
2	5	5	0.30×0.60	5.33	4.03	5.2×10^{-7}	5.33	4.03	5.2×10^{-7}
3	5	5	0.35×0.70	26.34	3.26	9.6×10^{-6}	27.74	3.38	6.5×10^{-6}
4	5	5	0.45×0.90	25.96	3.18	1.3×10^{-5}	27.41	3.30	8.8×10^{-6}
5	4	5	0.30×0.60	17.0	3.40	6.1×10^{-6}	19.33	3.55	3.3×10^{-6}
6	7	5	0.35×0.70	7.31	3.53	3.7×10^{-6}	8.0	3.61	2.7×10^{-6}
7	5	4	0.30×0.60	12.84	3.51	3.9×10^{-6}	13.67	3.62	2.6×10^{-6}
8	5	7	0.40×0.80	9.56	3.70	1.9×10^{-6}	9.56	3.70	1.9×10^{-6}
9	5	5	0.25×0.50	9.03	3.73	1.7×10^{-6}	9.40	3.83	1.1×10^{-6}
10	5	5	0.25×0.50	12.95	3.61	3.1×10^{-6}	13.53	3.71	1.8×10^{-6}

The length L [m] of the column is $L_2 = 3$, $L_3 = 9$, $L_4 = 12$ and $L_i = 6$ in all other cases. The number of storeys is 10, except for case 9 and 10 where it is 1 and 3, respectively.

In this example the LQI-criterion is active in all cases if the physical damage H_M is not included. If $H_M = 2.0 \cdot 10^6$ the optimal result is safer and therefore acceptable . The LQI-criterion is not active. In case 2 and 8 the constraint on the minimal reinforcement area is dominant. The resulting reliability indices per 50 years are within a narrow range from 3.2 to 4.0. The optimization having regard to the acceptability criterion derived from the Life Quality Index produces a rather uniform safety level in all study cases. The corresponding failure rates are comparatively low. As pointed out in [3] the optimum failure rates achievable by a design according to EUROCODES are by at least an order of magnitude smaller. Remember that $kN_{PE} = 1$ has been assumed which probably is unrealistically low. For this reason the effect of higher values of the number of fatalities $N_F = kN_{PE}$ in study case 1 on the failure rate is shown in figure 16. The LQI-data are the same as before. The solid line is derived from eq. (19) and the dashed lines show the optimal results determined from eq. (18). The latter are calculated with the cost for physical damage $H_M = 2.0 \cdot 10^6$ and $H_M = 4.0 \cdot 10^6$. The characteristics of the curves are the same as in the other study cases, if no additional constraint is active. As it can be seen the influence of the cost for physical damage diminishes with higer numbers of expected fatalities. The failure cost H_F are dependent on N_F whereas H_M is a given value. Therefore, the ratio between H_F and H_M tends to H_F with increasing N_F and the optimal curves coincide. In this example the optimal solution is only acceptable for a small number of expected fatalities. With increasing N_F the LQI-criterion becomes active. For $H_M = 2.0 \cdot 10^6$ the intersection between the optimal and the limit curves occurs already at $N_F \approx 2$. The limit derived from the LQI-criterion has to be taken for $N_F \gtrsim 2.5$. A higer value of damage cost in case of failure ($H_M = 4.0 \cdot 10^6$) forces the optimal solution towards acceptable failure rates until $N_F \lesssim 4.5$.

7. Example 6: Slender steel column with combined loading and intermittencies

The next example is a steel column with I-section and flange breadth μ_b, flange thickness μ_d and

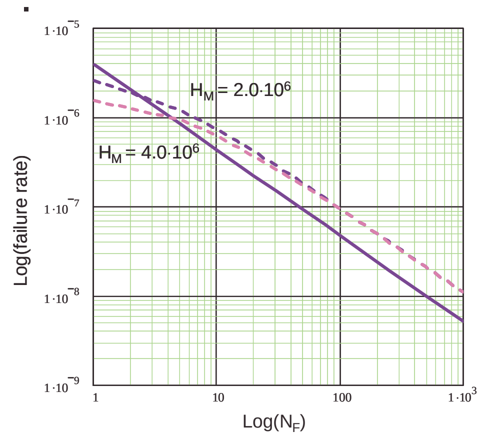

Figure 1: Risk-Consequence curves for various cost of physical damage in study case 1. Dashed lines correspond to optimal, the soild line to acceptable results.

height of steel profile μ_h. The design parameter μ_d and μ_h will be changed in the cost optimization. They are mean values of appropriate stochastic variables.

The objective function for systematic reconstruction and a stationary Poissonian renewal process is again given by eq. (6). $C(p) = C_0 + C_1\ (2\mu_b\mu_d + t\,\mu_h)\ s\ 8.2 * 10^{-9}$ are the construction cost of the steel column with length $s = 7500\ [mm]$, initial cost $C_0 = 100\ [CU]$ and $C_1 = 850\ [CU/m^3]$. The stochastic characteristics of the random variables are given by

Stochastic variable	Distr.	Mean/St.deviat.
Yield stress $F_S\ \left[\frac{N}{mm^2}\right]$	N	500/25
Load P_D [N]	N	400000/4000
Load P_1 [N]	N	800000/150000
Load P_2 [N]	LN	800000/150000
Load P_3 [N]	LN	800000/200000
Web thickness t [mm]	N	10/0.5
Flange Breadth B [mm]	LN	$\mu_b = 300$/0.1*μ_b
Flange Thickness D [mm]	LN	μ_d/0.1*μ_d
Height of profile H [mm]	LN	μ_h/0.1*μ_h
Initial deflection F_0 [mm]	N	m_{F_0}/σ_{F_0}
Youngs Modulus E $\left[\frac{N}{mm^2}\right]$	N	210000/4200

where the mean value m_{F_0} of F_0 is defined by

$$m_{F_0} = \frac{1}{20}\sqrt{\frac{0.5 \cdot B \cdot D \cdot H^{\ 2} + \frac{t \cdot H^{\ 3}}{12}}{2 \cdot B \cdot D + t \cdot H}} + \frac{s}{500}$$

and $\sigma_{F_0} = 0.3 \cdot m_{F_0}$. The limit state function in case of stability can be written approximately as

$$g(\mathbf{x}, \mathbf{p}) = F_S - P \cdot \left(\frac{1}{A_S} + \frac{F_0}{M_S} \cdot \frac{\epsilon_b}{\epsilon_b - P}\right) \tag{20}$$

where the auxiliary functions are defined by

$$\begin{array}{rcl} P & = & P_D + P_1 + P_2 + P_3 \\ A_S & = & 2BD \text{ (area of section)} \\ M_S & = & BDH \text{ (modulus of section)} \\ M_i & = & \frac{1}{2}BDH^2 \text{ (moment of inertia)} \\ \epsilon_b & = & \frac{\pi^2 E M_i}{s^2} \text{ (Euler buckling load)} \end{array}$$

Load P_1 is modelled as stationary differentiable Gaussian process with autocorrelation function $\rho_{ij}(\tau) = \exp(-\tau^2)$. The loads P_2 and P_3 are rectangular wave renewal processes with jump rates $\lambda_2 = 10.0$ and $\lambda_3 = 1.0\,[1/year]$. Also this mechanical model is an approximation which, however, fully serves the purpose.

The acceptability criterion is given by

$$\bigtriangledown_{\mathbf{p}} C(p) \geq -G_{\Delta\bar{\ell}} k N_{PE} \bigtriangledown_{\mathbf{p}} (\nu^+(\mathbf{p})) \tag{21}$$

and the LQI-data are $H_F = 6.0 \cdot 10^5$ and $G_{\Delta\bar{\ell}} k N_{PE} = 1.9 \cdot 10^6$ $(kN_{PE} = 1)$. The direct cost are $H_M = 3 \cdot C_0 = 300$. The objective function $Z(\mathbf{p})$ includes the cost H_F for the public with $b = 200$ and $\gamma = 0.0172$.

The interarrival-duration intensities are given for the loads P_1, P_2 and P_3 by $\rho_1 = 0.01$, $\rho_2 = 1.0$ and $\rho_3 = 100.0$. The results for an optimization of the height μ_h of the profile as design parameter with given values $\mu_d = 17.0$ and $\mu_b = 300$ are

Optimal μ_h	$\beta_i : PD+$	$P1$	$P2$	$P3$	$P1$	$P1$	$P2$	$P1+P2$
					$+P2$	$+P3$	$+P3$	$+P3$
348.6	13.6	7.7	7.1	6.2	5.3	4.94	4.93	3.6
Coincidence	4.9	4.9	4.9	4.9	4.9	4.9	4.9	4.9
Prob.	$\times 10^{-3}$	$\times 10^{-5}$	$\times 10^{-3}$	$\times 10^{-1}$	$\times 10^{-5}$	$\times 10^{-3}$	$\times 10^{-1}$	$\times 10^{-3}$
$\nu_i^{optimal}$	0	2.8	5.1	3.3	7.4	1.5	4.5	2.3
		$\times 10^{-14}$	$\times 10^{-12}$	$\times 10^{-10}$	$\times 10^{-7}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-3}$
Coinc. Prob.	0	1.4	2.5	1.6	3.6	7.5	2.2	1.1
$\times \nu_i^{optimal}$		$\times 10^{-18}$	$\times 10^{-14}$	$\times 10^{-10}$	$\times 10^{-11}$	$\times 10^{-9}$	$\times 10^{-6}$	$\times 10^{-5}$

The LQI-criterion is active at the optimal solution.

Without including the LQI-criterion the results are

Optimal μ_h	$\beta_i : PD+$	$P1$	$P2$	$P3$	$P1$	$P1$	$P2$	$P1+P2$
					$+P2$	$+P3$	$+P3$	$+P3$
346.1	13.5	7.7	7.1	6.1	5.3	4.91	4.90	3.5
$\nu_i^{optimal}$	0	4.0	6.7	3.9	9.1	1.8	5.3	2.6
		$\times 10^{-14}$	$\times 10^{-12}$	$\times 10^{-10}$	$\times 10^{-7}$	$\times 10^{-6}$	$\times 10^{-6}$	$\times 10^{-3}$
Coinc. Prob.	0	2.0	3.3	1.9	4.5	8.8	2.6	1.3
$\times \nu_i^{optimal}$		$\times 10^{-18}$	$\times 10^{-14}$	$\times 10^{-10}$	$\times 10^{-11}$	$\times 10^{-9}$	$\times 10^{-6}$	$\times 10^{-5}$

8. Example 7: Reinforced concrete column under combined loading and intermitencies

This example is also based on [3] . The design parameter is the reinforcement area A_s of the square shape $b \times h$ of the column cross section. A_s should satisfy the condition $0.003\, b\, h < a_s < 0.08\, b\, h$. The limit state function is expressed as the difference between the resistance bending moment M_R and the actual bending moment M about the cross section centre

$$g(\mathbf{x}, \mathbf{p}) = \xi_R\, M_R - \xi_E\, M \tag{22}$$

where the model uncertainties ξ_R and ξ_E are normally distributed with mean 1.1 and 1.0 respectively and coefficient of variation $V = 0.1$. Otherwise the same stochastic model for the strength parametes as in the foregoing reinforcement concrete column example is chosen. The loads are modelled as rectangular wave renewal processes with Gumbel distribution. It is distinguished between wind pressure with mean 0.00035 $[MN/m^2]$, standard deviation 0.00006 and jump rate $\lambda = 10$ $[year]$, long term imposed load with mean 0.0006 $[MN/m^2]$, standard deviation corresponding to the random point-in time distribution and jump rate $\lambda = 1/7$ $[1/year]$ and short term imposed load with mean 0.0002 $[MN/m^2]$, standard deviation corresponding to the distribution of 24 hours maximum and jump rate $\lambda = n$ $[1/year]$ ($n =$ number of storeys).

The interarrival-duration intensities are given for the wind pressure P_W, the long term imposed load P_L and the short term imposed load P_S by $\rho_W = 0.009$, $\rho_L = 3.0$ and $\rho_S = n/365$.

An objective function for systematic reconstruction and Poissonian failures is given by

$$Z(\mathbf{p}) = \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p}) + H_F)\frac{\nu^+(\mathbf{p})}{\gamma} \tag{23}$$

where $C(\mathbf{p}) = C_0 + (125\ [CU/m^3]\, b\, h + 4001\ [CU/m^3]\, A_s)\, L\ [m]$ with $C_0 = 300$ and L the length of the column.

The acceptability criterion can be written as

$$\bigtriangledown_{\mathbf{p}}\, C(\mathbf{p}) \geq -G_{\Delta\bar{\ell}} k N_{PE}\, \bigtriangledown_{\mathbf{p}} \left(\nu^+(\mathbf{p})\right) \tag{24}$$

Optimization will be performed for the public with benefit $b = 0.25$ and interest rate $\gamma = 0.0175$. The LQI-data are $H_F = 6.9 \cdot 10^5$ and $G_{\Delta\bar{\ell}} k N_{PE} = 1.9 \cdot 10^6$ with $kN_{PE} = 1.0$. The upper bound of the time variant failure probability in the stationary case is determined by $P_f(\mathbf{p}, t) = P_f(\mathbf{p}, t_1) + (t_2 - t_1)\nu^+(\mathbf{p})$. The equivalent beta values are derived from the upper bound solution. In the following table the optimal reinforcement area is calculated for various study cases, where the column length, the cross section, the distance to the surrounding columns and the number of storeys in the building is varied. A life time of 50 years is assumed.

Study case	Transversal distance [m]	Longitudinal distance [m]	Cross section $b \times h$ [m×m]	Optim. reinf. area $A_s 10^4$ [m^2]	equ. beta value
1	5	5	0.35×0.70	11.44	3.62
2	5	5	0.30×0.60	6.7	3.82
3	5	5	0.35×0.70	29.12	3.38
4	5	5	0.45×0.90	27.29	3.31
5	4	5	0.30×0.60	21.29	3.49
6	7	5	0.35×0.70	9.86	3.61
7	5	4	0.30×0.60	15.88	3.61
8	5	7	0.40×0.80	9.69	3.58
9	5	5	0.25×0.50	10.54	3.80
10	5	5	0.25×0.50	13.44	3.69

Study case	$\beta_i : PW, PS, PL = 0$	PS	PL	PW	$PS + PL$	$PS + PW$	$PL + PW$	$PS + PL + PW$
1	9.86	9.85	9.34	3.91	9.27	3.92	4.00	4.01
Coinc. Prob.	2.4×10^{-1}	7.2×10^{-1}	6.5×10^{-3}	2.2×10^{-3}	2.0×10^{-2}	6.5×10^{-3}	5.9×10^{-5}	1.8×10^{-4}
$\nu_i^{optimal}$	0	3.4×10^{-22}	5.7×10^{-24}	4.6×10^{-6}	4.2×10^{-22}	4.4×10^{-4}	7.6×10^{-6}	3.1×10^{-4}
Coinc. Prob. $\times \nu_i^{optimal}$	0	2.5×10^{-22}	3.7×10^{-26}	1.0×10^{-8}	8.2×10^{-24}	2.9×10^{-6}	4.5×10^{-10}	5.5×10^{-8}
2	7.61	7.53	7.02	4.15	6.96	4.12	3.94	3.91
3	9.87	9.87	9.85	3.69	9.85	3.70	3.76	3.77
$\nu_i^{optimal}$	0	2.7×10^{-22}	4.6×10^{-24}	8.7×10^{-6}	3.4×10^{-22}	8.5×10^{-4}	1.6×10^{-5}	6.7×10^{-4}
Coinc. Prob. $\times \nu_i^{optimal}$	0	1.9×10^{-22}	3.0×10^{-26}	1.9×10^{-8}	6.7×10^{-24}	5.5×10^{-6}	9.5×10^{-10}	1.2×10^{-7}
4	9.87	9.87	9.85	3.62	9.84	3.64	3.73	3.75
5	9.78	9.76	9.65	3.82	9.63	3.81	3.72	3.70
6	8.01	7.93	7.39	3.90	7.32	3.91	3.96	3.96
7	9.65	9.61	9.37	3.90	9.32	3.91	3.95	3.96
8	9.84	9.84	9.83	3.88	9.82	3.89	3.96	3.97
9	7.93	7.92	7.92	4.04	7.92	4.09	4.10	4.15
10	7.88	7.88	7.87	3.67	7.87	3.70	3.76	3.79

The length L [m] of the column is $L_2 = 3$, $L_3 = 9$, $L_4 = 12$ and $L_i = 6$ in all other cases. The number of storeys is 10, except for case 9 and 10 where it is 1 and 3, respectively.

In this example the LQI-criterion is active in all cases. The resulting reliability indices are within a narrow range from 3.3 to 3.8. The optimization having regard to the acceptability criterion derived from the Life Quality Index produces a rather uniform safety level in all study cases.

9. Example 8: Earthquake resistant design

A technically more involved example tries to establish a basis for the codification of design values for earthquake resistant design. It introduces so-called risk integrals. We follow closely the considerations in [8] but use slightly different data. In a seismic region Poissonian earthquakes occur with rate $\lambda = 2.9$ [1/year]. Magnitudes between $m_u = 4.0$ and $m_o = 7.5$ are considered. A truncated Weibull distribution (for maxima) has been found to model the data adequately

$$F_M(m) = \frac{\exp\left[-\left(\frac{m_o - m}{m_o - w}\right)^k\right] - \exp\left[-\left(\frac{m_o - m_u}{m_o - w}\right)^k\right]}{1 - \exp\left[-\left(\frac{m_o - m_u}{m_o - w}\right)^k\right]} \tag{25}$$

with $w = 4.35$ and $k = 8.11$. These data are characteristic for an area with medium to high seismicity. With the attenuation law

$$a = h(m,r) = b_1 \exp(b_2 m)(r^2 + 7.3^2)^{-1/2} \exp(-b_3 r) = \exp(b_2 m) b(r) \tag{26}$$

where $b_1 = 0.0955g, b_2 = 0.573, b_3 = 0.00587$, one determines the density of peak ground acceleration as

$$f_A(a,r) = \frac{k \frac{(m_o - h^{-1}(a,r))^{k-1}}{(m_o - w)^k} \exp\left[-\left(\frac{(m_o - h^{-1}(a,r))}{m_o - w}\right)^k\right] \frac{dh^{-1}(a,r)}{da}}{1 - \exp\left[-\left(\frac{m_o - m_u}{m_o - w}\right)^k\right]} \tag{27}$$

with $m_u \leq h^{-1}(a,r) = \frac{1}{b_2} \ln(\frac{a}{b(r)}) \leq m_o$. Possible epicentres are uniformly distributed around the site in a radius of $r_{\max} = 200$ km. Hence, the density of peak ground acceleration is:

$$f_A(a) = \int_0^{r_{\max}} f_A(a,r) \frac{2r}{r_{\max}^2} dr \tag{28}$$

Peak ground acceleration then varies with a coefficient of variation of $V_A = 1.55$. The maximum responses given peak ground acceleration vary log-normally with coefficient of variation of $V_S = 0.60$. A simplified limit state function then is

$$g(\mathbf{X}) = R - KSAE \leq 0 \tag{29}$$

for shear resistance versus shear demand. Herein, R is a log-normal resistance with $V_R \approx 0.2$, K contains all system-specific properties and is, without loss of generality, assumed to equal unity, S is the log-normal variability in the (elastic) spectral enhancement factor with mean $m_S = 1$ and A is peak ground acceleration. The systematic frequency-dependent part of S must be taken into account in K. E is the log-normal error in relation (26) with mean 1 and coefficient of variation $V_E \approx 0.6$. The conditional failure probability (fragility curve) is

$$P_f(p \mid a) = \Phi\left(-\frac{\ln\left\{\frac{p}{K \cdot m_S \cdot a \cdot m_E}\sqrt{\frac{(1+V_S^2)(1+V_E^2)}{1+V_R^2}}\right\}}{\sqrt{\ln\left((1+V_R^2)(1+V_S^2)(1+V_E^2)\right)}}\right) \tag{30}$$

with $p = m_R$ the design parameter because $m_S m_E = 1.0$. The objective function (without benefit term) for systematic reconstruction is

$$\begin{aligned} Z(p) &= C(p) f(N_{PE}) + \\ &+ E_A\left[\begin{array}{c} \left(C_R(p,a)(1 - P_f(p \mid a)) f(N_{PE}) \frac{\lambda}{\gamma}\right) + \\ + ((C(p) + H_0 + H_M(a)) f(N_{PE}) + H_F(a)) \frac{\lambda P_f(p|a)}{\gamma} \end{array}\right] \end{aligned} \tag{31}$$

The acceptability criterion eq. (**??**) to be used as a constraint for eq. (31) correspondingly reads:

$$\frac{d}{dp}\left(C_0 + C_1 p^\delta\right) \geq -E_A\left[G_{\Delta\bar{\ell}} \frac{1}{2}(1 - \exp(-0.25a)) N_{PE} \frac{d}{dp}\left(\lambda P_f(p,a)\right)\right] \tag{32}$$

The following widely verified relationship between peak ground acceleration and MSK-intensity $log(a) = 0.31MSK - 2.5$ is assumed. We distinguish between normal damage to the building during an earthquake and building collapse. Construction, retrofitting, loss of business and physical damage cost are slightly underproportional to the occupation rate of a unit in a residential building so that we choose $f(N_{PE}) = (N_{PE}/3)^{0.8}$. $C(p) = \left(C_0 + C_1 p^\delta\right)$ is the construction cost, $C_R(p,a) =$

$C(p)(1 - \exp(-0.25a))$ is the cost of retrofitting taking account of the fact that retrofitting cost approach the cost of complete reconstruction for larger a, $H_M(a) = H_M a^{0.4}$ is the physical damage cost. The physical damage term includes infrastructure losses for large accelerations. Indirect cost such as loss of business is approximated by $H_0 = \ell C_0$. The estimation of human losses is difficult. They also depend on a as people are increasingly trapped at collapse. Immediate death then has probability 0.3 to 0.5 or more but some 10 to 20% die later in hospital. This leads to $H_F(a) = SLSC\,\frac{1}{2}(1 - \exp(-0.25a))N_{PE}$ for the compensation cost for loss of human life. Note that the factor $\frac{1}{2}(1 - \exp(-0.25a))$ replaces the constant k in eq. (7). The constant 0.25 in these relationships appears to vary with building type and material. The concavity of the function with respect to a implies convexity with respect to MSK in agreement with estimates in the literature. It is certainly only a rough approximation. Furthermore, we have $C_0 = 10^6, C_1 = 3 \cdot 10^4, \delta = 1.1$, $H_M = 5 \cdot 10^5$, $\gamma = 0.02$, $N_{PE} = 3$. It is beyond the scope of this paper to discuss all these special choices in detail but they are essentially in line with the findings in [1] and other sources in the literature. The damage term in eq. (31) is conditional on a. The expectation operation removes the condition. The damage term is also called risk integral. The table below collects typical data for three different socio-economic levels.

Socio-economic level	High	Medium	Low
GDP	23500	6500	1500
q	0.145	0.17	0.20
e	77	65	55
C_Δ	35	50	65
ℓ	3.62	1.0	0.23

An additional FORM/SORM-analysis can then determine the design values a^* corresponding to p^* and some other results of interest in the following results table.

Socio-economic level	High	Medium	Low
p^*	4.80	4.00	3.54
$h(p^*) = \lambda P_f(p^*)$ (FORM)	$3.7 \cdot 10^{-4}$	$6.3 \cdot 10^{-4}$	$8.8 \cdot 10^{-4}$
Return period $\frac{1}{h(p^*)}$	2700	1600	1100
a^*	1.03	0.97	0.84
$a^* s^* \varepsilon^*$	3.11	2.76	2.43
$C(p^*)/C_0$	1.17	1.14	1.12

The design values of the accelerations a^* have a return period of about 120 years. Roughly the same design accelerations for all socio-economic levels indicate that the quantity $E_A[\lambda P_f(p \mid a)]$ decays very slowly with a. The values of $a^* s^* \varepsilon^* < p^*$ are also given. Surprisingly, the acceptance criterion eq. (32) is not active and produces values $p_{\lim}$ of approximately 1.5, 1.0 and 0.5, respectively. These values imply an order of magnitude larger failure rates than the optimal solution. This example is somewhat extreme because the loading side varies very much. Large changes in the values of p^* result in rather small changes in the failure rates. This explains why the differences between the different socio-economic climates are relatively small. Also, in contrary to the previous examples, both construction cost and damage cost have been referred to a residential unit and are roughly proportional to N_F. Therefore, the effect of varying N_F is insignificant and the failure rates in the second line of the results table are the individual risks due to earthquakes in that region.

10. Conclusions

The examples show the application of the reliability-oriented cost-benefit optimization for structural components in the so-called one-level approach. An acceptability criterium as constraint of the optimization problem ensures that the optimal structure is also a sufficiently safe structure. Acceptibly safe structures are determined by the LQI-approach. Optimization is performed from the public's and the owner's point of view. The acceptability criterion becomes active especially if the owner uses higher discount rates. The optimal results are significantly influenced by the stochastic model chosen for the relevant uncertainties and the coefficient of variation of the resistance variables and of the load processes.

Example 2 is proposed to be taken as standard as it can demonstrate the effect of many factors. The few examples with realistic physical and stochastic models are well supporting the findings in example 2. The computed acceptable risk-consequence curves show an almost perfect dependence of the type $r_{acc}(N_F) = \frac{K}{N_F}$ with $K = 10^{-3}$ (case with large variabilities and/or costly safety measures), 10^{-4} (standard case) and 10^{-5} (for small variabilities and inexpensive safety measures). K has dimension [1/year]. Almost all examples with somehow realistic physical and/or stochastic models and realistic parameters demonstrate that optimization leads generally to safer structures and the constant K falls in between 10^{-4} and 10^{-5} for $N_F = 1$.

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