

Examples on the Application of the *LQ* Criterion

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1 Introduction

In the following three examples are provided on the use of the *LQI* criterion for the assessment of the conformity of decision alternatives with societal preferences for investments into life saving activities. In a more usual language this is also understood as an assessment of whether decision alternatives are societal acceptable. The examples consider the following decision situations:

- Continuous decision alternatives
- Discrete decision alternatives
- Combinations of discrete decision alternatives

and thereby aim to cover the spectrum of the most usually occurring decision situations in engineering design and assessment.

2 Design of a Structural Member

Here we consider a typical engineering decision situation where the cross-sectional dimension of a steel bar under tension loading is to be optimized and assessed for acceptability. The cross-sectional dimension of relevance, i.e. the cross-sectional area A is continuous decision parameter. The failure event of the bar is modeled by the event that the yield stress of the bar is exceeded by the load effect due to the external loading, here assumed associated with an extreme event of a variable load S . The annual extreme load effect modeled by a Gumbel distribution with an expected value $\mu_s = 9.50 kN$ and a standard deviation $\sigma_s = 1.50 kN$. The resistance R of the steel bar is given through the cross-sectional area A and the yield stress of the steel f_y . The yield stress of the steel is modeled by a Log-normal distribution with expected value $\mu_R = 260 MPa$ and standard deviation $\sigma_R = 18.2 MPa$.

The annual probability of failure conditional on the cross-sectional area of the steel bar can thus be calculated as:

$$P_f(A) = \int_0^{\infty} F_R(x|A) f_s(x) dx \quad (1)$$

The *LQI* criterion can be written as:

$$\frac{dP_f(A)}{dA} - \frac{q}{C_{\Delta, d=2\%, a} N_{PE} k g} \frac{dC_y(A, t, \gamma)}{dA} < 0 \quad (2)$$

In order to proceed it is now necessary to fix the various parameters of Equation (2). It is thus assumed that the conditional probability of mortality k given the event of failure of the steel bar is equal to 1. Furthermore it is assumed that 12 persons are exposed to the failure event.

The costs per mm^2 steel cross sectional area are assumed to be equal to 5000 CHF. Assuming that in the case of failure the steel bar will be systematically renewed to its original condition the net present value of the construction costs (including future renewal costs) may be represented by:

$$C_y(A, t, \gamma) = \left(C_{y,1}^*(A) + C_{y,2}^*(A) \cdot (1 - (1 + \gamma)) \cdot (\ln(1 + \gamma))^{-1} \cdot P_f(A) \right) \cdot t^{-1} \quad (3)$$

where $C_{y,1}^*(A), C_{y,2}^*(A)$ correspond to the construction costs and the renewal costs of the steel bar respectively; here assumed identical equal to $C_y^*(A)$. γ corresponds to a constant annual discounting rate.

Table 1: Parameters required for the decision optimization.

g	Gross Domestic Product per capita	35931 CHF
$C_{\Delta, d=2\%, a}$	Demographic constant	18.9
q	-	0.175
k	Conditional probability of mortality given failure	1
N_{PE}	Number of exposed persons	12
$C_y^*(A)$	Construction costs	5000 · A CHF
b	Benefit of structure	$1.2 \cdot 10^4$ CHF / yr
C_o	Compensation costs	1.8 Mio. CHF
C_U	Cleaning costs	$3 \cdot 10^4$ CHF
γ	Interest rate	2%
t	Time period	100 yr

In Table 1 a summary is provided of the parameters required for the structural optimization and assessment of acceptance.

The optimal structural design is identified through the following object function:

$$Z(A, t, \gamma) = b \cdot \frac{1 - (1 + \gamma)^{-t}}{\ln(1 + \gamma)} - C_y^*(A) - P_f(A) \cdot \left(C_y^*(A) + k N_{PE} C_o + C_U \right) \cdot \frac{1 - (1 + \gamma)^{-t}}{\ln(1 + \gamma)} \quad (4)$$

where it is assumed that the benefit $b(A)$ associated with the structure is constant equal to $1.2 \cdot 10^4$ CHF / yr. The failure costs include besides the renewal costs $C_y^*(A)$ the costs of cleaning C_v as well as compensations for lost lives C_o . The annual constant discounting rate is set equal to 2% and a time period t of 100 years is chosen. Since a long time period is considered in this example, the discount rate is chosen according to the long term economic growth rate. Rackwitz (2005) provides a discussion on discount rates.

The optimization of the object function in Equation (4) and the assessment of the *LQI* criterion from Equation (2) is illustrated in Figure 1 A) - 1 D).

Based on the *LQI* criterion it is found that cross-sectional areas smaller than 93 mm^2 cannot be accepted. This corresponds to an annual probability of failure equal to $P_f = 5.15 \cdot 10^{-6} [\text{yr}^{-1}]$, and construction costs equal to $4.65 \cdot 10^3$ CHF yr^{-1} .

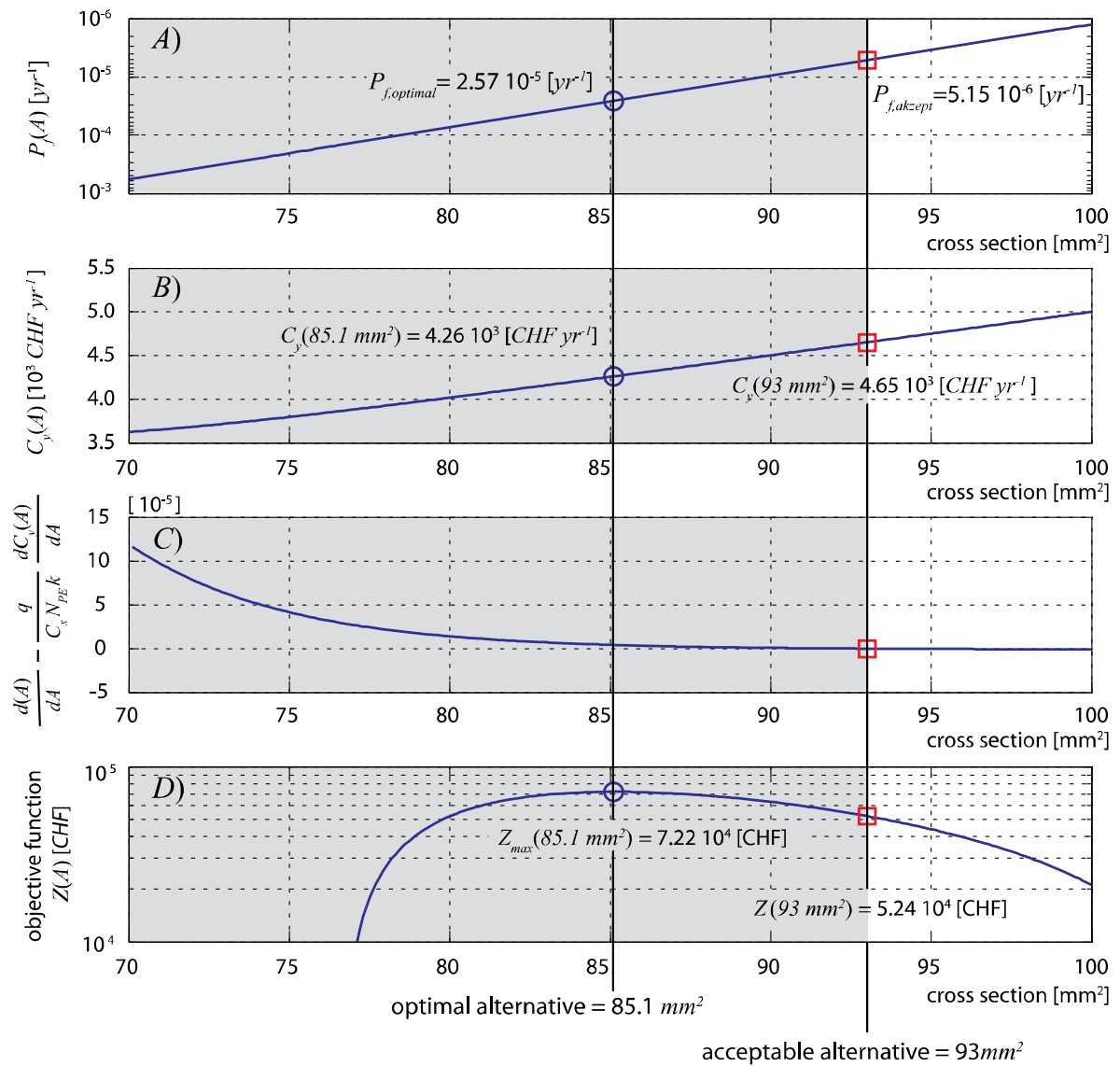


Figure 1: Summary of the results.

It is seen that the object function does not attain its maximum in the acceptable region of decision alternatives. The societal preference for investments into life saving in this case requires that more is invested into life saving.

3 Protection Against Natural Hazards

In this example the identification of decision alternatives for the protection against natural hazards is considered. It is assumed that a specific location is considered where people are exposed to natural hazards through some activity which yields a benefit for society; e.g. a roadway segment exposed to rock-fall or avalanche hazards. Furthermore, it is assumed that the decision to establish the roadway segment in the first place in the way it was done was optimal and in coherency with societal preferences in regard to investments into life saving activities at the time.

The problem complex investigated in the further is related to whether additional protective measures which have become available since the roadway segment was established should be pursued from the perspective of societal acceptance as well as from the perspective of economical optimization.

At the location of interest the natural hazard of relevance occurs with an annual rate equal to $\nu(p_0) = 0.035$. The conditional probability of mortality $k(p_0)$ given the event of the natural hazard is assumed equal to 0.40 and the expected value of the number of exposed persons is $N_{PE}(p_0) = 2.0$.

The investigated five different decision alternatives are denoted by p_i , $i = 1, 2, \dots, 5$. Depending on the chosen decision alternative the event of the natural hazard is reduced to $\nu(p_i)$ and the number of exposed persons is changed to N_{PE, p_i} .

An analysis of the effect and costs of the different decision alternatives has resulted in the results which are summarized in Table 2.

Table 2: Parameters for the different decision alternatives.

Option	ν_{p_i} [yr ⁻¹]	$C_y^*(p_i)$ [CHF]	$C_y(p_i, t, \gamma)$ [CHF yr ⁻¹]	C_{U, p_i} [CHF]	N_{PE, p_i} [CHF]	k_{p_i} [CHF]	$\nu_{p_i}^*$ [yr ⁻¹]
p_0	$3.5 \cdot 10^{-2}$	-	-	$0.5 \cdot 10^3$	2	0.4	-
p_1	$2.0 \cdot 10^{-2}$	$5.0 \cdot 10^3$	$9.4 \cdot 10^2$	$1.0 \cdot 10^3$	4	0.8	$8.7 \cdot 10^{-3}$
p_2	$2.0 \cdot 10^{-3}$	$1.5 \cdot 10^5$	$1.6 \cdot 10^3$	$1.5 \cdot 10^3$	5	0.3	$1.8 \cdot 10^{-2}$
p_3	$5.0 \cdot 10^{-3}$	$3.0 \cdot 10^6$	$3.7 \cdot 10^4$	$1.9 \cdot 10^3$	10	0.4	$4.6 \cdot 10^{-3}$
p_4	$5.0 \cdot 10^{-4}$	$2.0 \cdot 10^5$	$2.0 \cdot 10^3$	$2.0 \cdot 10^3$	8	0.3	$1.1 \cdot 10^{-2}$
p_5	$1.0 \cdot 10^{-4}$	$1.0 \cdot 10^6$	$1.0 \cdot 10^4$	$7.0 \cdot 10^3$	10	0.2	$1.3 \cdot 10^{-2}$

Based on the *LQI* criterion it is now assessed which of the different decision alternatives are acceptable. The *LQI* criterion yields:

$$v_0(p_i) \leq \frac{N_{PE}(p_0)k(p_0)v(p_0) + \frac{C_y(p_i, t, \gamma) q}{C_{\Delta, d=2\%, a} g}}{N_{PE}(p_i)k(p_i)} = v^*(p_i) \quad (5)$$

With the annual costs of a given decision alternative given as:

$$C_y(p_i, t, \gamma) = \left(C_y^*(p_i) + C_y^*(p_i) \cdot \left(1 - (1 + \gamma)^{-t}\right) \cdot (\ln(1 + \gamma))^{-1} \cdot v(p_i) \right) \cdot t^{-1} \quad (6)$$

And parameters as given in Table 1 and Table 2.

From Equation (5) it is found that as long as $v^*(p_i)$ larger is than $v(p_i)$ the decision alternative i is conform with societal preferences for life saving and must thus be accepted. For the present example this is the case for decision alternatives p_2, p_4 and p_5 .

As it is seen that three of the decision alternatives are acceptable the question is which of these are the optimal, considering a maximization of life-cycle benefits. The object function to be optimized can be written as:

$$Z(p_i, t, \gamma) = B(p_i, t, \gamma) - C_y^*(p_i) - R(p_i, t, \gamma) \quad (7)$$

in which the benefit of the individual decision alternatives $B(p_i, t, \gamma)$ are given through the reduction in life risks and can be determines as:

$$B(p_i, t, \gamma) = \left(v_0 k_0 N_{PE,0} - v(p_i) k_{p_i} N_{PE, p_i} \right) \cdot C_o \cdot \frac{1 - (1 + \gamma)^{-t}}{\ln(1 + \gamma)} \quad (8)$$

The remaining risk is calculated by:

$$R(p_i, t, \gamma) = v(p_i) \left(C_y(p_i) + C_U(p_i) \right) \cdot \frac{1 - (1 + \gamma)^{-t}}{\ln(1 + \gamma)} \quad (9)$$

In Figure 2 a *LQI*-diagram is shown (Nathwani et al. (1997)). The values on the *y-axis* are proportional to the change in the expected number of fatalities $N_{PE, pi} k_{pi} \Delta v_{pi}$ with $\Delta v_{pi} = v_{pi} - v_{p_0}$. The proportionality factor for the *y-axis* is g . The values on the *x-axis* are proportional to the annual costs of a given decision alternative $\Delta C_y(p_i)$. The proportionality factor for the *x-axis* is the $q C_x^{-1}$. The shaded area in Figure 2 illustrates the region where decisions are not acceptable, i.e. $N_{PE, pi} k_{pi} \Delta v_{pi} < \Delta C_y(p_i) q C_x^{-1}$. The results for the investigated decision alternatives $p_1 - p_5$ are plotted in Figure 2.

From the *LQI*-diagram it is clearly seen that the decision alternatives p_1 and p_3 are not acceptable. Only the options p_5, p_4, p_2 can be accepted. *Table 3* summarizes the results of the assessment of decision alternatives.

Table 3: Overview of assessment result for the investigated decision alternatives.

Option	$v(p_i)$	$v^*(p_i)$	$B(p_i, t, \gamma)$	$C_y^*(p_i)$	$R(p_i, t, \gamma)$	$Z(p_i, t, \gamma)$
p_1	$2.0 \cdot 10^{-2}$	$8.7 \cdot 10^{-3}$	$-2.82 \cdot 10^5$	$5.0 \cdot 10^3$	$4.44 \cdot 10^4$	$-3.76 \cdot 10^5$
p_2	$2.0 \cdot 10^{-3}$	$1.8 \cdot 10^{-2}$	$1.96 \cdot 10^5$	$1.5 \cdot 10^5$	$1.32 \cdot 10^4$	$+3.27 \cdot 10^4$
p_3	$5.0 \cdot 10^{-3}$	$4.1 \cdot 10^{-3}$	$6.27 \cdot 10^4$	$1.0 \cdot 10^6$	$6.53 \cdot 10^5$	$-3.59 \cdot 10^6$
p_4	$5.0 \cdot 10^{-4}$	$1.1 \cdot 10^{-2}$	$2.1 \cdot 10^5$	$2.0 \cdot 10^5$	$4.4 \cdot 10^3$	$+5.58 \cdot 10^2$
p_5	$1.0 \cdot 10^{-4}$	$1.3 \cdot 10^{-2}$	$2.18 \cdot 10^5$	$1.0 \cdot 10^6$	$4.83 \cdot 10^3$	$-7.87 \cdot 10^5$

It is interesting to note that decision alternative p_2 maximizes the life-cycle benefit even though the decision alternative p_5 yields a lower life risk.

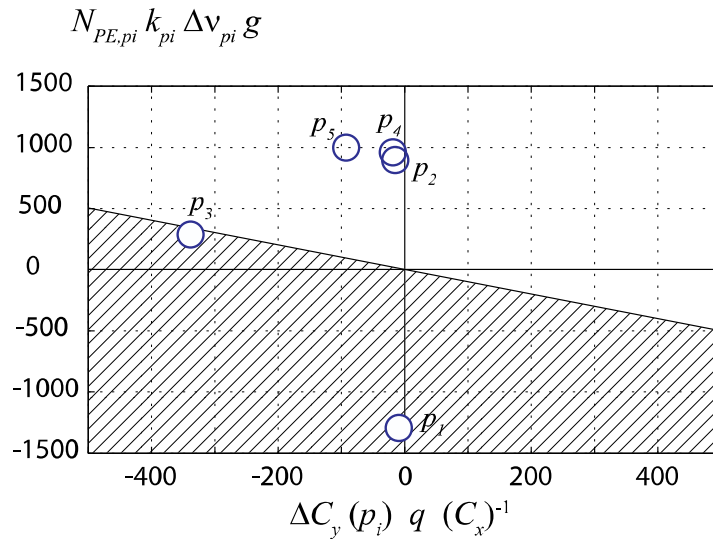


Figure 2: *LQI*-Diagram for the five considered decision alternatives.

4 Protection Against Natural Hazards

As a final example consider the assessment of decision alternatives comprised through packages of discrete decision alternatives. This would correspond to the foregoing example but where now new decision alternatives are constructed by combination of the previous. It is assumed that the initial life risk is given as $\lambda_{p_0} = N_{PE, p_0} \cdot k_{p_0} \cdot v_{p_0} = 3.5 \cdot 10^{-3} \text{ yr}^{-1}$. Five different decision alternatives to reduce the risks are to be combined and assessed in regard to optimality and acceptability. The annual costs C_{y, p_i} and the risk reductions $\Delta \lambda_{p_i}$ corresponding to the different alternatives are summarized in Table 4.

It is further assumed that the different decision alternatives can be freely combined and that their combined effect on risk reduction as well as their costs can be assessed by simple addition. As a first step the alternatives are ranked in accordance with their efficiency. The

most efficient decision alternative is p_1 , as this represents the largest ration between $\Delta\lambda_{p_i}$ and C_{y,p_i} . The next most efficient decision alternative is p_2 . The ranked list of decision alternatives is provided in column 5 of Table 4.

Table 4: Input parameters for the assessment of packages of measures.

Alternative	$\Delta\lambda_{p_i}$	C_{y,p_i}	$\Delta\lambda_{p_i}/C_{y,p_i}$	Rank
	[yr^{-1}]	[$CHF yr^{-1}$]	[CHF^{-1}]	[–]
p_1	$2.0 \cdot 10^{-2}$	$3 \cdot 10^3$	$6.7 \cdot 10^{-6}$	1
p_2	$2.0 \cdot 10^{-3}$	$2 \cdot 10^3$	$1.0 \cdot 10^{-6}$	3
p_3	$8.0 \cdot 10^{-3}$	$7 \cdot 10^3$	$1.1 \cdot 10^{-6}$	2
p_4	$4.0 \cdot 10^{-3}$	$9 \cdot 10^3$	$4.4 \cdot 10^{-7}$	4
p_5	$8.0 \cdot 10^{-4}$	$1 \cdot 10^4$	$8.0 \cdot 10^{-8}$	5

The combinations of packages of risk reducing measures to be investigated can now simply be established through the simple successive addition of alternatives in accordance with the rank number. In this way first combination $p_1 + p_2$ is assessed, thereafter $p_1 + p_2 + p_3$, etc. Additional risk reducing measures, following the ranked list, should be pursued as long as the acceptance criterion for packages is fulfilled:

$$\Delta C_y(p_i | \theta_i) - \frac{g}{q} C_x N_{PE}(p_i | \theta_i) k(p_i | \theta_i) \Delta v(p_i | \theta_i) \geq 0$$

$$\Leftrightarrow \Delta C_y(p_i | \theta_i) - \frac{g}{q} C_x \Delta \lambda(p_i | \theta_i) \geq 0 \tag{10}$$

$$\theta_i = (p_1, p_2, \dots, p_i)$$

Table 5: Calculation of the acceptability of the combination.

Combination θ_i	Alternative	$\Delta\lambda_{p_i}$	$\Delta C_y(p_i \theta_i)$	$\Delta C_y(p_i \theta_i) - \frac{g}{q} C_x \Delta \lambda(p_i \theta_i)$
	[–]	[yr^{-1}]	[yr^{-1}]	
p_1	p_1	$2.0 \cdot 10^{-2}$	$2.0 \cdot 10^{-2}$	$7.5 \cdot 10^4$
p_1, p_3	p_3	$8.0 \cdot 10^{-3}$	$8.0 \cdot 10^{-3}$	$2.4 \cdot 10^3$
p_1, p_3, p_2	p_2	$2.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$5.8 \cdot 10^3$
p_1, p_3, p_2, p_4	p_4	$4.0 \cdot 10^{-3}$	$4.0 \cdot 10^{-3}$	$6.5 \cdot 10^3$
p_1, p_3, p_2, p_4, p_5	p_5	$8.0 \cdot 10^{-4}$	$8.0 \cdot 10^{-4}$	$-6.9 \cdot 10^4$

The result is given in Table 5. For the present example this implies that all decision alternatives $p_1 - p_4$ should be implemented. For p_5 the acceptance criterion is no longer fulfilled. This may be visualized by the indifference curve for the *LQI* shown in Figure 3.

The dashed lines in Figure 3 illustrates the *LQI* indifference curves. Any decision which simply corresponds to expenditures (or gains) and life savings (losses) for which the corresponding *LQI* remains the same are *LQI* indifferent; corresponds to a translation in the Figure along the *LQI* indifference curves. The direction indicated on the *LQI* indifference curves show in which direction the *LQI* increments positively. The actual value of the *LQI* is simply read from the given location in the plot. The most efficient risk reducing measure with the highest efficiency yields the largest increment in *LQI*, i.e. alternative p_1 , the second most efficient alternative is p_3 . The incremental benefit of the implementation of additional risk reducing measures is seen to approach the gradient of the *LQI* indifference curve. For the last decision alternative it is seen that the increment is seen to be smaller the gradient of the *LQI* indifference curve and this alternative should thus not be implemented. If this alternative (i.e. p_5) would be implemented this would lead to a societal loss; the benefit is smaller than the expenditure. The maximal societal benefit achievable from the implementation of the packages of decision alternatives is illustrated through d_{max} , see Figure 3.

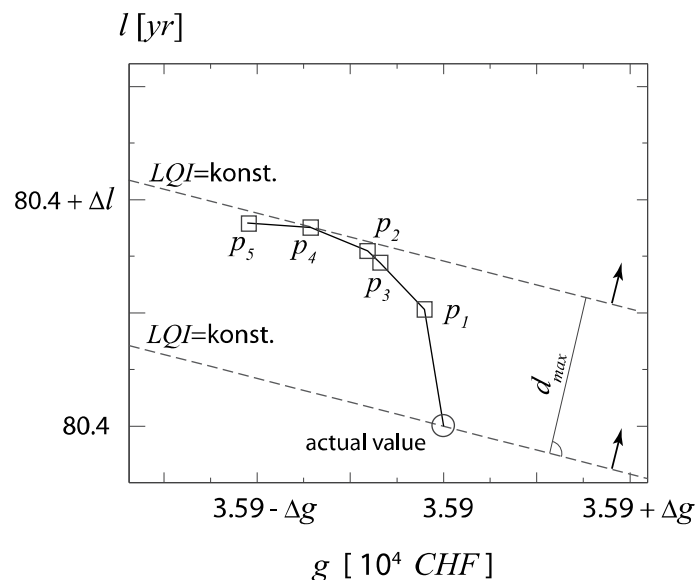


Figure 3: *LQI* indifference curves and *LQI* increase associated with the combined decision alternatives p_1, p_1, \dots, p_5 .

The foregoing only addressed the assessment of the conformity of decision alternatives with societal preferences in regard to life saving investments. The assessment of the economical optimality of the investigated packages should of course also be undertaken. This is however, not addressed in this example but follows straightforwardly following the same principle as for individual discrete decision alternatives, see also the preceding example.

References

- Nathwani, J. S., Lind, N. C. & Pandey, M. D. (1997) *Affordable Safety by Choice: The Life Quality Method*, Waterloo, Ontario, Canada, Institute for Risk Research, University of Waterloo.
- Rackwitz, R. (2005) *The Philosophy Behind the Life Quality Index and Empirical Verifications. Memorandum to JCSS*. Munich, Germany.